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A perturbation approach to transport in discrete ratchet systems

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Abstract. We consider a perturbation approach for the calculation of current in a discrete ratchet system under the influence of outer noise. In the lowest order (quadratic in potential changes) the current contains contributions of two different types. One of them is of the first order in the transition rate changes and depends only on the amplitude of the noise and the other corresponds to second-order perturbation and depends on the noise's temporal correlation function. The two types of terms correspond to two different mechanisms of rectification. The explicit calculations are performed for a three-level model system and are compared with the numerical solution of the corresponding master equation. The comparison shows that the perturbation approach works well even for outer fields comparable with the ratchet potential.

1. Introduction

The kinetics of so-called driven ratchet systems have received much attention during the last five years since its relation to the problem of cellular motors and pumps was understood, see [1–4] for review. Thus, a variety of different continuous and discrete models was introduced [5–19], most of them sharing the main features of the initial model of [5]. This model corresponds to a particle in a static periodic saw-tooth potential lacking inversion symmetry. Application of a time-dependent (periodic or stochastic) outer field with a zero mean causes the directed motion of the particle. Although the formulation of the problem is very simple, only a few variants of stochastically driven ratchet equations can be solved analytically. Thus the formulation of a generally applicable approximate method could be of great value. In what follows we consider a perturbative approach to the problem. We confine ourselves to discrete models which are probably more relevant in the description of biophysical processes, due to the well-defined structure of molecular states involved.

We stress here that perturbation theory is not merely an approximate computation method but is an important tool for the classification of types of response of a nonlinear system to (small) outer fields. Thus, the nonlinear properties of a system are described by expanding the response function in a power series in an applied field, just as is done in nonlinear optics (see e.g. [20]). The rectification of weak outer fields normally appears as a second-order effect. As we proceed to show, a natural perturbation expansion for a discrete ratchet system is an expansion in powers of the transition rates rather than the outer field itself. The transition rates are nonlinear (exponential) functions of the outer field, so that the second-order effect in a field stems both from the first- and the second-order contributions in transition rates. Thus, discrete ratchets exhibit, in general, two different mechanisms of rectification, which could

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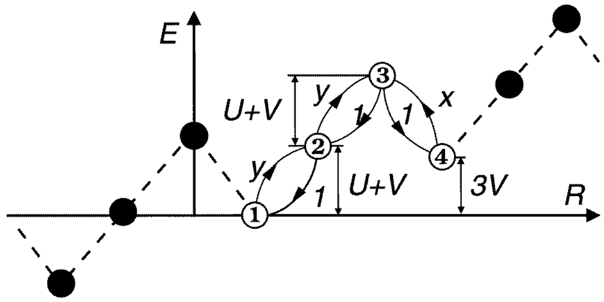

Figure 1. The structure of the array used in the considerations.

be described as due to direct and induced (parametric) nonlinearity. The first mechanism is responsible for the rectification of fields with short correlation times. Another one, involving the occupancies' redistribution, is manifest only if the outer field is correlated over times which are comparable with characteristic relaxation times of the system, i.e. when the system tends to an adiabatic regime. These findings allow us to express the rectified current in the ratchet through the power spectrum of the noise and to directly connect the rectification properties with the characteristic timescales of the system and the stochastic field ('noise'), a question addressed in [17] for a special case of a cycling stochastic force.

2. General considerations

We begin with a master equation for continuous-time random walks of a particle on a one-dimensional lattice with a time-dependent ratchet-type potential [13, 16], as given in figure 1. Let $p_k(t)$ be the probability of finding a particle at site k at time t, and w_{ij} denote the (time-dependent) transition rates between neighbouring sites i and j. Then the master equation reads:

$$\dot{p}_k(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} p_k(t) = w_{k-1,n} p_{k-1}(t) + w_{k+1,k} p_{k+1}(t) - (w_{k,k-1} + w_{k,k+1}) p_k(t). \tag{1}$$

The 'elementary engine' (one period of the saw-tooth potential) consists of n sites ($n = 3$ in figure 1) and is connected to the right and to the left to site 1 of the next elementary engine (site 4 in figure 1) and to site n of the previous one. The transition rates w_{ij} depend on the potential differences between the different sites of the ratchet. At constant temperature T, the rates in equation (1) obey a detailed balance condition; using the Metropolis procedure we can take all downhill rates to be equal, and set in this case $w_{ij} = w_0$. The uphill rates are then:

$$w_{ij}(t) = w_0 \exp\left(-\frac{U_{ij}(t)}{k_B T}\right). \tag{2}$$

Here $U_{ij}(t)$ is the energy difference between neighbouring sites and k_B is the Boltzmann constant. In what follows we keep T constant and set $w_0 = 1$ and $k_B T = 1$. In figure 1 the potential difference U_{ij} is $U + V$ between the pairs of sites $(1, 2)$ and $(2, 3)$ and $-2U + V$ between (3,4). Changes in V modulate the overall steepness of the potential, whereas changes in U determine the height of the saw-teeth. The regime of operation corresponding to changes in U is quoted as a flashing regime, the changes in V correspond to a rocked one. The potential differences $U_{ij}(t) = U_{ij}^0 + U_{ij}(t)$ include the static ratchet potential U_{ij}^0 and the time-dependent outer potential difference $U_{ij}(t)$ (outer field), which will be taken to be small.

In operation, the probabilities $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ in each spatial period of n sites are equal to those of any other one, $(mn+1, mn+2, \ldots, nm+n)$, for all m. Then, at a given time t, the state of the engine is characterized by $(p_1, p_2, \ldots, p_n)$ only. This leads to a system of n ordinary differential equations of the form:

$$\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} = \boldsymbol{W}(t)\boldsymbol{p}(t) \tag{3}$$

with $\boldsymbol{p}(t_0)$ being the vector of initial conditions, $|\boldsymbol{p}(t_0)| = \sum_i p_i(t_0) = 1$. Here $\boldsymbol{W}$ is a matrix of coefficients w_{ij} (taking into account cyclic boundary conditions which are implied by the translational invariance of the system: $W_{1,n} = w_{n+1,n}$ and $W_{n,1} = w_{n,n+1}$).

Distinguishing between the static ratchet potential and the outer field, we put $\boldsymbol{W}(t) = \boldsymbol{W}_0 + \delta\boldsymbol{W}(t)$, where $\boldsymbol{W}_0$ is time independent (having the elements w_{ij}^0 which satisfy a detailed-balance condition in the absence of the outer field) and $\delta\boldsymbol{W}$ is a matrix of $\delta w_{ij}(t) = w_{ij}(t) - w_{ij}^0$. Since we are interested in the effects of the second order in the outer field and since the dependence of w_{ij} on this field is given by equation (2), the perturbation expansion up to the second order in $\delta\boldsymbol{W}$ is sufficient.

Let us first review some properties of $\boldsymbol{W}(t)$, following from general requirements to stochastic matrices (see e.g. [21]). Thus, the probability conservation implies that the sum of the elements of each column is zero:

$$\sum_i w_{ij} = 0 \tag{4}$$

so that $\det \boldsymbol{W}(t) = 0$. This property is shared by both matrices $\boldsymbol{W}_0$ and $\delta\boldsymbol{W}$. Let the eigenvalues of $\boldsymbol{W}_0$ be $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$. The values $\lambda_2, \ldots, \lambda_n$ have negative real parts connected with the inverse relaxation times of the system. Let $\boldsymbol{p}^1, \boldsymbol{p}^2, \ldots, \boldsymbol{p}^n$ denote the corresponding (right) eigenvectors: $\boldsymbol{W}_0\boldsymbol{p}^i = \lambda_i\boldsymbol{p}^i$. The right eigenvector $\boldsymbol{p}^1$ corresponding to the zero eigenvalue of the unperturbed matrix describes the stationary state of the system: $\mathrm{d}\boldsymbol{p}^1/\mathrm{d}t = \boldsymbol{W}_0\boldsymbol{p}^1 = 0$. Let us suppose that no special symmetries are present and thus the eigenvalues of the matrix $\boldsymbol{W}_0$ are nondegenerate. In this case $\boldsymbol{W}_0$ can be diagonalized by a linear transformation whose matrix $\boldsymbol{L}$ is built up from $\boldsymbol{p}^i$ taken as columns: $\boldsymbol{L} = (\boldsymbol{p}^1, \boldsymbol{p}^2, \ldots, \boldsymbol{p}^n)$. Under a linear transformation we have: $\boldsymbol{M} = \boldsymbol{L}^{-1}\boldsymbol{W}(t)\boldsymbol{L} = \boldsymbol{M}_0 + \delta\boldsymbol{M}$, with $\boldsymbol{M}_0 = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_n)$ and $\delta\boldsymbol{M} = \boldsymbol{L}^{-1}\delta\boldsymbol{W}(t)\boldsymbol{L}$. Here $\mathrm{diag}(\ldots)$ denotes a diagonal matrix with the elements listed in brackets. Now, let $\boldsymbol{q}^i$ be the left eigenvector of $\boldsymbol{W}_0$ corresponding to the eigenvalue λ_i: $\boldsymbol{q}^i\boldsymbol{W}_0 = \lambda_i\boldsymbol{q}^i$, so that $\boldsymbol{q}^1\boldsymbol{W}_0 = 0$. From equation (4) it then follows that all elements of $\boldsymbol{q}^1$ are equal. Note that the matrix $\boldsymbol{L}^{-1}$ is built from $\boldsymbol{q}^i$ taken as lines since the diagonalization procedure implies that $\boldsymbol{L}^{-1}\boldsymbol{W}_0 = \boldsymbol{M}_0\boldsymbol{L}^{-1}$. Thus the first line of $\boldsymbol{L}^{-1}$ is proportional to $(1, 1, \ldots, 1)$ and all elements in the first line in $\delta\boldsymbol{M}$ vanish:

$$(\delta\boldsymbol{M})_{1l} = \sum_{j,k}(\boldsymbol{L}^{-1})_{1j}\delta w_{jk}\boldsymbol{L}_{kl} \propto \sum_k \boldsymbol{L}_{kl}\sum_j \delta w_{jk} = 0. \tag{5}$$

We note that for an unbiased ratchet system the stationary state corresponds to the thermodynamic equilibrium. Moreover, we confine ourselves to the overdamped system where the relaxation to equilibrium in an unperturbed system takes place aperiodically, without oscillations. In this case all eigenvalues other than λ_1 are real and negative and can be associated with inverse relaxation times of the system, $\lambda_i = -\tau_i^{-1}$.

Let $\boldsymbol{e}^i = \boldsymbol{L}^{-1}\boldsymbol{p}^i$ be the eigenvectors of $\boldsymbol{M}_0$. In the basis of vectors $\boldsymbol{e}^i$ we can expand the time-dependent solution near the equilibrium one, i.e. look for a solution $\boldsymbol{x}(t) = \boldsymbol{L}^{-1}\boldsymbol{p}(t)$ in the form:

$$\boldsymbol{x}(t) = \boldsymbol{e}^1 + \boldsymbol{x}_1(t) + \boldsymbol{x}_2(t) + \cdots \tag{6}$$

where $\delta \boldsymbol{x}_i$ can be determined recurrently by

$$\dot{\boldsymbol{x}}_i - \boldsymbol{M}_0 \boldsymbol{x}_i = \delta \boldsymbol{M} \boldsymbol{x}_{i-1} \tag{7}$$

with $\boldsymbol{x}_0 = \boldsymbol{e}^1$. Let us imagine that the outer field was switched on at $t = t_0$ and that for $t < t_0$ the system was in equilibrium. This assumption implies the initial conditions $\boldsymbol{x}_i = 0$ for all $i > 0$. Introducing $\boldsymbol{E}(t) = \exp(\int_0^t \boldsymbol{M}_0(t')\,\mathrm{d}t') = \mathrm{diag}(1, \mathrm{e}^{\lambda_1 t}, \ldots, \mathrm{e}^{\lambda_n t})$ we can write the solution of equation (7) in the form $\boldsymbol{x}^i(t) = \int_{t_0}^t \mathrm{d}t'\, \boldsymbol{E}(t-t')\delta \boldsymbol{M}(t)\boldsymbol{x}^{i-1}$. In the limit $t_0 \to -\infty$ one then obtains:

$$\boldsymbol{x}_1 = \int_{-\infty}^t \mathrm{d}t'\, \boldsymbol{E}(t-t')\delta \boldsymbol{M}(t)\boldsymbol{e}^0 \tag{8}$$

$$\boldsymbol{x}_2 = \int_{-\infty}^t \mathrm{d}t'\, \boldsymbol{E}(t-t')\delta \boldsymbol{M}(t)\boldsymbol{x}_1(t) = \int_{-\infty}^t \mathrm{d}t'\, \boldsymbol{E}(t-t')\delta \boldsymbol{M}(t') \times \int_{-\infty}^{t'} \mathrm{d}t'\, \boldsymbol{E}(t'-t'')\delta \boldsymbol{M}(t'')\boldsymbol{e}^1 \tag{9}$$

etc. Now one can return to a site-basis by a back-transformation $\boldsymbol{p} = \boldsymbol{L}^{-1}\boldsymbol{x}$.

In order to calculate the current through the bond connecting the sites i and $i+1$ of the system we introduce the current operator, corresponding to a scalar product of $\boldsymbol{p}$ with a vector $\boldsymbol{J} = (0, \ldots, w_{i,i+1}, -w_{i+1,i}, \ldots, 0)$ whose only nonzero elements are the ones on places i and $i+1$. Thus, $j(t) = \boldsymbol{J}(t)\boldsymbol{p}(t)$. Note that again $\boldsymbol{J}(t) = \boldsymbol{J}_0 + \delta \boldsymbol{J}(t)$ and that $\boldsymbol{J}_0 \boldsymbol{p}^1 = 0$ in the case when the stationary state is assumed to be an equilibrium one.

Thus, in the second order of perturbation theory one has: $j(t) = \boldsymbol{J}(\boldsymbol{t})\boldsymbol{p}(\boldsymbol{t}) = \delta \boldsymbol{J}(t)\boldsymbol{p}_0 + \boldsymbol{J}_0 \boldsymbol{p}_1(t) + \delta \boldsymbol{J}(t)\boldsymbol{p}_1(t) + \boldsymbol{J}_0 \boldsymbol{p}_2(t)$. Averaging this expression over different realizations of noise one gets:

$$J = \overline{\boldsymbol{J}(\boldsymbol{t})\boldsymbol{p}(\boldsymbol{t})} = \overline{\delta \boldsymbol{J}(t)}\boldsymbol{p}_0 + \boldsymbol{J}_0 \overline{\boldsymbol{p}_1(t)} + \overline{\delta \boldsymbol{J}(t)\boldsymbol{p}_1(t)} + \boldsymbol{J}_0 \overline{\boldsymbol{p}_2(t)}. \tag{10}$$

The averaging over the realizations of the stochastic outer field is commutative both with time integration and with linear transformation $\boldsymbol{L}$. Thus,

$$\overline{\boldsymbol{p}_1} = \boldsymbol{L}^{-1}\overline{\boldsymbol{x}_1} = \boldsymbol{L}^{-1}\left(\int_{-\infty}^t \mathrm{d}t'\, \boldsymbol{E}(t-t')\right)\overline{\delta \boldsymbol{M}}\boldsymbol{e}^1 = \boldsymbol{L}^{-1}\boldsymbol{T}(t)\overline{\delta \boldsymbol{M}}\boldsymbol{e}^1 \tag{11}$$

where $\boldsymbol{T}(t) = \mathrm{diag}(t, \lambda_i^{-1})$. Note that although $\boldsymbol{T}_{11}(t) = t$, no terms proportional to time are actually contained in equation (11) since, according to equation (5), $(\delta \boldsymbol{M}\boldsymbol{e}^0)_1$ vanishes identically. On the other hand

$$\overline{\boldsymbol{x}_2} = \left\langle \int_{-\infty}^t \mathrm{d}t'\, \boldsymbol{E}(t-t')\delta \boldsymbol{M}(t') \int_{-\infty}^{t'} \mathrm{d}t'\, \boldsymbol{E}(t'-t'')\delta \boldsymbol{M}(t'')\boldsymbol{e}^1 \right\rangle = \int_{-\infty}^t \mathrm{d}t'\, \boldsymbol{E}(t-t') \int_{-\infty}^{t'} \mathrm{d}t''\langle \delta \boldsymbol{M}(t')\boldsymbol{E}(t'-t'')\delta \boldsymbol{M}(t'')\rangle \boldsymbol{e}^1. \tag{12}$$

Now, the matrix $\boldsymbol{F}(t', t'') = \langle \delta \boldsymbol{M}(t')\boldsymbol{E}(t'-t'')\delta \boldsymbol{M}(t'')\rangle$ has the elements

$$F_{ij}(t_1, t_2) = \left\langle \sum_k \delta M_{ik}(t_1)\mathrm{e}^{\lambda_k(t_1-t_2)}\delta M_{kj}(t_2) \right\rangle = \sum_k \mathrm{e}^{\lambda_k(t_1-t_2)}\langle \delta M_{ik}(t_1)\delta M_{kj}(t_2)\rangle. \tag{13}$$

(Equation (5) leads again to the fact that all elements proportional to t actually vanish.) For a system driven by a homogeneous random process the correlation functions $C_{ik,kj}(t_1, t_2) = \langle \delta M_{ik}(t_1)\delta M_{kj}(t_2)\rangle$ depend only on $t_2 - t_1$. The second integral in equation (12) is now independent of t' and thus the first integration can be immediately performed:

$$\overline{\boldsymbol{x}_2} = \boldsymbol{T}(t)\int_{-\infty}^0 \mathrm{d}\tau\, \boldsymbol{F}(\tau)\boldsymbol{e}^1 \tag{14}$$

from which $\overline{\boldsymbol{p}_2}$ follows by linear transformation. Note that since $\boldsymbol{x}$ does not contain elements which grow with time, the elements of $\overline{\boldsymbol{p}^2}$ are linear combinations of the integrals of the form $\int_{-\infty}^{0} \mathrm{d}\tau\, \mathrm{e}^{-\lambda_k \tau} C(\tau)$. Changing the variable of integration from τ to $-\tau$ one notices that these terms correspond to the Laplace transforms of the correlation functions of δM, with the Laplace variable corresponding to the eigenvalues (inverse relaxation times) of the system. The same procedure can be applied when calculating $\overline{\delta \boldsymbol{J}(t)\boldsymbol{p}_1(t)}$ where the integrals of a similar structure appear.

We note here that the mean current, equation (10), can be considered as a sum of two contributions of a different form: $J = J_1 + J_2$. $J_1 = \overline{\delta \boldsymbol{J}(t)}\boldsymbol{p}_0 + \boldsymbol{J}_0\overline{\boldsymbol{p}_1(t)}$ is of first order in δw and thus only depends on one-time characteristics of the transition rates. $J_2 = \overline{\delta \boldsymbol{J}(t)\boldsymbol{p}_1(t)} + \boldsymbol{J}^0\overline{\boldsymbol{p}_1(t)}$ is of second order and depends on the two-time correlation properties of δw_{ij}. Remembering now that $w_{ij}(t) = \exp(-U_{ij} - \delta U_{ij}(t))$, so that $w_{ij} = \exp(-U_{ij})$ and $\delta w_{ij} = w_{ij}(-\delta U_{ij} + \delta U_{ij}^2/2 + \cdots)$ we see that $\overline{\delta w_{ij}} = \overline{\delta U_{ij}^2}/2$ and that the two-time correlators of the transition rates can be expressed through the corresponding correlator of outer field δU. Thus, the first-order terms are all proportional to the mean square value of the outer potential and do not depend on its correlation properties. On the other hand the second-order terms depend explicitly on the two-time correlation functions of the outer field.

3. Results for a model system

Let us return to a model system as shown in figure 1, i.e. to a three-level ratchet. For this system one has:

$$\boldsymbol{W} = \begin{pmatrix} -(x(t) + y(t)) & 1 & 1 \\ y(t) & -(1 + y(t)) & 1 \\ x(t) & y(t) & -2 \end{pmatrix} \tag{15}$$

where

$$y(t) = w_{12} = w_{23} = \exp\left(-\frac{U_{12}(t)}{k_B T}\right) \tag{16}$$

and

$$x(t) = w_{43} = \exp\left(-\frac{U_{31}(t)}{k_B T}\right). \tag{17}$$

We suppose both $x(t)$ and $y(t)$ to be smaller than unity. Let x and y be the values of $x(t)$ and $y(t)$ in the absence of an outer potential, so that

$$\boldsymbol{W}_0 = \begin{pmatrix} -(x + y) & 1 & 1 \\ y & -(1 + y) & 1 \\ x & y & -2 \end{pmatrix} \tag{18}$$

and

$$\delta \boldsymbol{W}(t) = \begin{pmatrix} -(\delta x(t) + \delta y(t)) & 0 & 0 \\ \delta y(t) & -\delta y(t) & 0 \\ \delta x(t) & \delta y(t) & 0 \end{pmatrix}. \tag{19}$$

The eigenvalues of $\boldsymbol{W}_0$ are $\lambda_1 = 0$, $\lambda_2 = -(1 + x + y)$ and $\lambda_2 = -(2 + y)$. The corresponding right eigenvectors are

$$\boldsymbol{p}^1 = \frac{1}{(2 + y)(1 + x + y)} \begin{pmatrix} 2 + y \\ 2y + x \\ x + xy + y^2 \end{pmatrix}$$
$$\boldsymbol{p}^2 = \begin{pmatrix} 1 - x \\ y - 1 \\ x - y \end{pmatrix} \quad \text{and} \quad \boldsymbol{p}^3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{20}$$

(the first one is taken to be normalized to unity). From this the form of the matrices $\boldsymbol{L}$ and $\boldsymbol{L}^{-1}$ follow. In what follows we calculate the average current through the 34 (i.e. 31) bond, and thus take $\boldsymbol{J}(t) = (-x(t), 0, 1)$. The straightforward calculations following the scheme outlined in the previous section then give

$$J_1 = -\frac{1+y+y^2}{(2+y)(1+x+y)^2}\overline{\delta x} + \frac{x^2+3y^2+xy^2+6xy+3x+4y}{(2+y)^2(1+x+y)^2}\overline{\delta y} \tag{21}$$

and

$$\begin{aligned} J_2 = {} & \frac{1+y+y^2}{(2+y)(1+x+y)^2}[F_{xx}(1+x+y)+F_{xy}(1+x+y)] \\ & + \frac{y^2+y+xy-x^2-x^2y-1}{(2+y)(1+x+y)^2(x-1)}[F_{yx}(1+x+y)+F_{yy}(1+x+y)] \\ & + \frac{3x-x^2-xy-y^2}{(2+y)^2(1+x+y)(x-1)}F_{yy}(2+y) \\ & - \frac{y-1}{(2+y)(1+x+y)(x-1)}F_{yx}(2+y) \end{aligned} \tag{22}$$

where $F_{ab}(z) = \int_0^t \mathrm{e}^{-zt} C_{ab}(t)\,\mathrm{d}t$. Here $C_{ab}(t)$ are the correlation functions of the type $C_{ab}(t) = \langle \delta a(0)\delta b(t)\rangle$, with a and b being x or y. Turning to an unbiased ratchet, which would be in equilibrium for a zero outer field, one has $x = y^2$, so that all coefficients in equation (21), (22) are functions of y only:

$$J_1 = -f(y)(\overline{\delta x} - 2y\overline{\delta y}) \tag{23}$$

and

$$\begin{aligned} J_2 = f(y)\Bigg\{ & F_{xx}(A) + F_{xy}(A) + \frac{1-y-y^2}{1+y}[F_{yx}(A)+F_{yy}(A)] \\ & + \frac{1}{1+y}[y^2F_{yy}(B) - F_{yx}(B)]\Bigg\} \end{aligned} \tag{24}$$

with $f(y) = (2+y)^{-1}(1+y+y^2)^{-1}$. On the other hand the relations between $\overline{\delta x}$ and $\overline{\delta y}$ and the relations between different F-functions depend on the mode of operation.

Let us first consider the rocked ratchet, for which the values of U_{ij} are modulated by the additive outer potential difference $V(t)$ applied between each pair of the neighbouring sites, see [13, 16]. In this case one has

$$y(t) = \exp(-U - V) \tag{25}$$

and

$$x(t) = \exp(-2U + V) \tag{26}$$

with $\overline{V(t)} = 0$. One readily gets $y = \exp(-U)$ and $\delta y = y[-V(t) + V^2(t)/2 + \cdots]$. Moreover, $x = \exp(-2U) = y^2$ and $\delta x = y^2[V(t) + V^2(t)/2 + \cdots]$. Substitution of these values into equation (23) gives

$$J_1 = y^2 f(y) v^2/2 \tag{27}$$

where we have introduced the variance of the outer potential $v^2 = \overline{V^2(t)}$. Note that J_1 is always positive. Calculating the F-functions in the lowest order in V we get: $F_{xx}(z) = x^2F_{VV}(z) = y^4F_{VV}(z)$, $F_{xy}(z) = F_{yx}(z) = -y^3F_{VV}(z)$ and $F_{yy}(z) = y^2F_{VV}(z)$, so that

$$J_2 = f(y)\left[y^2\frac{2y^3-3y+1}{1+y}F_{VV}(1+y+y^2) + y^3\frac{1-y}{1+y}F_{VV}(2+y)\right] \tag{28}$$

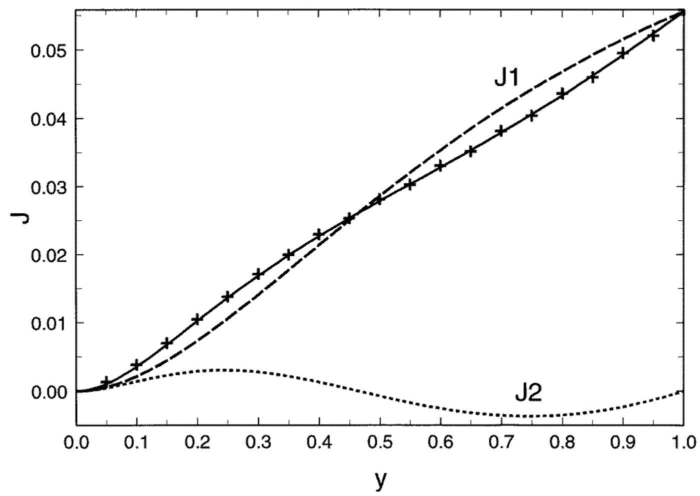

Figure 2. The behaviour of the overall current in a rocked ratchet and its first- and second-order contributions as functions of y for the exponentially correlated noise with $\tau = 1$, see text for details. Plotted are J (solid curve), $J1$ (dashed curve) and $J2$ (dotted curve). The crosses denote the results of numerical simulation.

which is also quadratic in the potential's amplitude. For example, for a dichotomic noise with the correlation function

$$C_{VV}(t) = v^2 \exp(-|t|/\tau) \tag{29}$$

one gets: $F_{VV}(z) = \frac{v^2\tau}{1+z\tau}$. The behaviour of $J_1(y)$, $J_2(y, \tau)$ and of the overall current J as functions of y is shown in figure 2 for $\tau = 1$. The functions $J = \overline{J}/v^2$, $J1 = J_1/v^2$ and $J2 = J_2/v^2$ are plotted. Note that in the case considered the largest contribution to the overall current stems from the first-order term that does not depend on the correlation properties of the noise and which is a monotonously growing function of y. On the other hand, the second-order term shows the reversal behaviour, but is too small to cause the overall current reversal. We also note that the current in a rocked system stays finite both in the cases $\tau \to 0$ (white noise) and $\tau \to \infty$ (adiabatic regime).

To assess the numerical accuracy of the perturbation approach we compare the behaviour of J as a function of y with the one obtained through the numerical solution of equation (3) with $\boldsymbol{W}(t)$ given by equations (15), (25) and (26) using the Eulerian scheme with time step $\Delta t = 0.01$. The outer field of the constant absolute value v is chosen to simulate a Markovian dichotomic process with correlation function, equation (29). Thus, with probability $\Delta t/\tau$ per integration step, the sign of $V(t)$ is chosen anew. The averaging is performed over the time $T = 10^5\tau$. The results of numerical integration are shown in figure 2 as crosses for the amplitude of the outer field $v = 0.1$; the coincidence of the results is excellent. We also note that even for v as large as one, the difference between the numerical results and the perturbation predictions does not exceed 10% within the range $0 < y < 1$. The same good quality of approximation was demonstrated when we considered another dichotomic process, in which the change of sign of the outer field takes place with probability $\Delta t/\tau$ per integration step. For this process $F_{VV}(z) = \frac{v^2\tau(z\tau+3)}{(z\tau+2)(z\tau+1)}$.

Let us now turn to the flashing case, for which

$$y(t) = \exp(-U_0)\left[1 - \delta U(t) + \frac{\delta U^2(t)}{2} + \cdots\right] \tag{30}$$

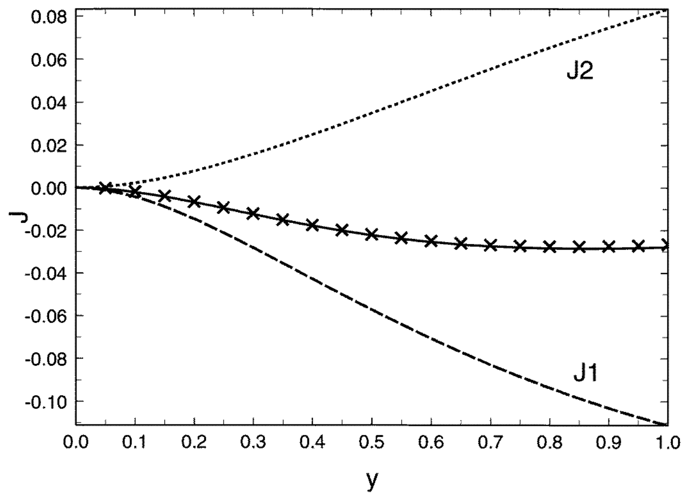

Figure 3. Same as in figure 2, now for a flashing case (see text for details). Note that both $J1$ and $J2$ are monotonous and have opposite signs.

and

$$x(t) = \exp(-2U_0)[1 - 2\delta U(t) + 2\delta U^2(t) + \cdots] \tag{31}$$

so that $\delta y(t) = y[-\delta U(t)+\delta U^2(t)/2+\cdots]$ and $\delta x(t) = y^2[-2\delta U(t)+2\delta U^2(t)+\cdots]$. For this case we get $F_{xx}(z) = 4y^4 F_{UU}(z)$, $F_{xy}(z) = F_{yx}(z) = 2y^3 F_{UU}(z)$ and $F_{yy}(z) = y^2 F_{UU}(z)$. Equation (23) now gives:

$$J_1 = -y^2 f(y) v^2 \tag{32}$$

with $v^2 = \overline{\delta U^2(t)}$. This expression for J_1 is always negative. The expression for J_2 reads

$$J_2 = f(y)\left[y^2 \frac{2y^3 + 3y^2 + 3y + 1}{1 + y} F_{UU}(1 + y + y^2) - y^3 \frac{2 + y}{1 + y} F_{UU}(2 + y)\right]. \tag{33}$$

The behaviour of $J_1(y)$, $J_2(y, \tau)$ and of the overall current J for the field with $C_{UU}(t) = v^2 \exp(-|t|/\tau)$ (with $\tau = 1$) is shown in figure 3. Again, the crosses denote the results of the numerical integration for $v = 0.1$. Note that in this case J is a (small) difference of two large contributions. We must note that in this case the approximation is more critical to the amplitude of the outer field, leading for $v = 1$ to deviations as large as 30%. Considering the τ-dependence of the current we find that for very slowly varying noise ($\tau \to \infty$), when $F_{UU}(z) \to v^2 z^{-1}$, the contributions from $J_1(y)$ and $J_2(y, \tau)$ compensate each other exactly, so that in the adiabatic case the flashing system produces no current. This is no surprise, since such a system with multiplicative noise is, in this case, always in equilibrium.

We stress here that in both cases, the rocked and the flashing one, the system can easily rectify the noise which is almost white. We note that the impossibility to rectify white noise by a continuous ratchet (the disappearance of a ratchet effect in a high-frequency limit [18]) is to no extent a consequence of general thermodynamics, but a rather special property of a continuous rocked model, whose internal noise is also white, cf [19], where a white-noise-driven flashing ratchet is considered.

Let us now turn to the case of the coloured noise correlated over very long times. In figure 4 we plot the behaviour of the current as a function of y for rocked and flashing models, where we take $C_{VV}(t)$ to follow a power law: $C_{VV}(t) = 1/(1+|t|/\tau)^\gamma$ with $\tau = 1$ and $\gamma = 0.5$

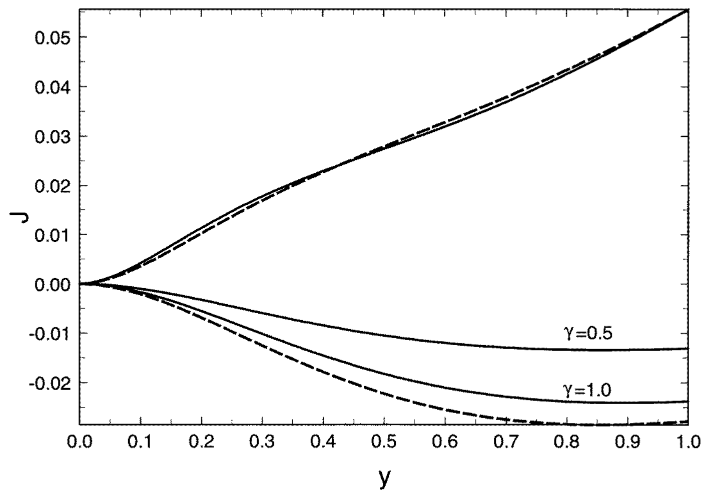

Figure 4. The y-dependence of $J(y)$ for ratchets under power-law-correlated noise (solid curves) with $\gamma = 0.5$ and $\gamma = 1$ and under exponentially correlated noise (dashed lines, identical with the J-dependencies of figures 2 and 3, respectively). The upper curves correspond to the rocked case, and the lower curves to the flashing case. The curves corresponding to different power-law-correlated noises for a rocked ratchet are indistinguishable on the scales of the plot.

and $\gamma = 1$. The last case corresponds to the type of noise considered in [22]. The functions $F_{VV}(z)$ are given respectively by $F_{VV}(z) = \sqrt{\pi/z}\mathrm{e}^z\mathrm{erfc}(\sqrt{\pi z})$ and $F_{VV}(z) = -\mathrm{e}^z\mathrm{Ei}(-z)$, see [23]. The behaviour of $j(y)$ in these cases (shown in solid curves) is compared with those for an exponentially correlated noise (dashed curves). Note that all differences are of a quantitative nature. For a flashing system the values of the current under different types of the noise differ considerably, but the qualitative behaviour of the curves remains the same. For a rocked system these differences are marginal (since J_2 depending on the temporal correlations is small compared with J_1); the curves for the two types of power-law correlated noise are indistinguishable on the scales of the plot.

4. Conclusions

We have considered a perturbative approach to the calculation of current in discrete ratchet systems. We have shown that in the lowest (quadratic) order in the outer field the current consists of two contributions, describing two different mechanisms of rectification. One of them appears as a first-order perturbation in the transition rate changes and is independent of the temporal correlations of the noise. This mechanism is responsible for the rectification of noise whose correlation time is small compared with the characteristic relaxation times of a system. It can be considered as a manifestation of a nonlinearity in transition rates. Another contribution stems from the second-order perturbation and thus depends on the temporal correlation function of the outer field. This contribution describes a typically parametric effect. We continued with model calculations performed for a three-level discrete model of [13, 16] in rocked and flashing regimes. The comparison of the perturbation approach with the results of direct numerical simulations shows that the accuracy of our approximation is excellent for moderate amplitudes of the field. The behaviour of the system under the noise showing power-law correlations was also discussed.

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